

There are non homotopic framed homotopies of long knots

Thomas Fiedler

March 10, 2009

Abstract

Let \mathcal{M} be the space of all, including singular, long knots in 3-space and for which a fixed projection into the plane is an immersion. Let $cl(\Sigma_{iness}^{(1)})$ be the closure of the union of all singular knots in \mathcal{M} with exactly one ordinary double point and such that the two resolutions represent the same (non singular) knot type. We call $\Sigma_{iness}^{(1)}$ the *inessential walls* and we call $\mathcal{M}_{ess} = \mathcal{M} \setminus cl(\Sigma_{iness}^{(1)})$ the *essential diagram space*.

We construct a non trivial class in $H^1(\mathcal{M}_{ess}; \mathbb{Z}[A, A^{-1}])$ by an extension of the Kauffman bracket. This implies in particular that there are loops in \mathcal{M}_{ess} which consist of regular isotopies of knots together with crossing changings and which are not contractible in \mathcal{M}_{ess} (leading to the title of the paper).

We conjecture that our construction gives rise to a new knot polynomial for knots of unknotting number one.

1

1 Introduction and results

The study of knot spaces consists mainly of the study of spaces of non singular knots (compare [2], [4], [10], [3]). In this paper we change the point of view: we construct a 1-cocycle for some space which includes singular knots too.

¹2000 *Mathematics Subject Classification*: 57M25. *Key words and phrases*: classical and singular links, discriminant, knot polynomials, state models

This 1-cocycle is identical zero on all loops which consist only of non-singular knots.

The space of all (possibly singular long) knots was introduced and studied by Vassiliev in the pioniering work [15]. It is the space of all differentiable maps $f : \mathbb{R} \rightarrow \mathbb{R}^3$ which agree with $x \rightarrow (x, 0, 0)$ outside of $[-1, 1]$. This is a contractible space (compare [15]), let us call it \mathcal{F} . We identify then as usual a knot with its image $f(\mathbb{R})$. A knot is *non-singular* if $f(\mathbb{R})$ is a smooth submanifold, otherwise it is called *singular*. Knots are oriented from the left to the right.

The singular knots form the discriminant Σ of the space of all (long) knots. It has a natural stratification. The complement of the discriminant are the *chambers*. They correspond to the (non-singular) knot types. Vassiliev has introduced a filtration on the cohomology of the chambers. The 0-dimensional part are the well-known Vassiliev knot invariants (for all this compare [1], [15], [16]).

Instead of Vassiliev's knot space \mathcal{F} we will study the space \mathcal{M} of long framed knots. We fixe a plane in 3-space which contains the x -axes and we fixe an orthogonal projection pr of the 3-space into this plane. \mathcal{M} is the space of all those (possibly singular) knots for which the restriction of pr on the knots is an immersion. The chambers of \mathcal{M} consist of knot diagrams up to isotopy and up to Reidemeister moves of type II and III (this is called usually *regular isotopy* of non singular knots). The points in \mathcal{M} project to immersed long planar curves. It is easy to see that the connected components of \mathcal{M} are in 1-1 correspondence with the regular homotopy classes of immersed long planar curves (standard at infinity). It is well known that the latter are in 1-1 correspondence with their Whitney index n (i.e. the degree of the Gauss map). One easily sees that \mathcal{M} is a disjoint union of contractible spaces (numbered by the Whitney index).

The strata of codimension one $\Sigma^{(1)}$ of the discriminant are called the *walls*. They correspond to the knots which have exactly one ordinary double point as the only singularity. An ordinary double point of an oriented knot can be resolved in two different ways (compare Fig. 1).

Definition 1 *A wall is called inessential if the two adjacent chambers coincide. The union of all inessential walls is denoted by $\Sigma_{iness}^{(1)}$. The space $\mathcal{M} \setminus cl(\Sigma_{iness}^{(1)})$ is called the essential diagram space and denoted by \mathcal{M}_{ess} . A generic path in \mathcal{M}_{ess} is called a framed homotopy.*

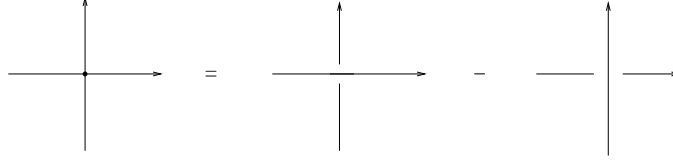


Figure 1: Vassiliev's skein relation



Figure 2: not framed homotopic knots?

We could associate to each connected component of \mathcal{M} a stratified space in the following way: the chambers of \mathcal{M} correspond to vertices. Each stratum $S^{(1)}$ in $\Sigma^{(1)}$ corresponds to an edge connecting the vertices corresponding to chambers adjacent to $S^{(1)}$. Each stratum $S^{(2)}$ in $\Sigma^{(2)}$ gives rise to a 2-cell glued to all edges and vertices corresponding to those strata $S^{(1)}$ and those chambers which are adjacent to $S^{(2)}$. And we can go on by gluing cells of dimension n corresponding to strata of codimension n .

The resulting space is not a CW-complex because it is not locally finite. (It is well known that there are e.g. infinitely many knot types of unknotting number one.) However, this space is evidently still simply connected.

Let us now consider the closure $cl(\Sigma_{iness}^{(1)})$ in a component of \mathcal{M} of the union of all inessential walls (i.e. we add all adjacent strata of higher codimension). This space is a rather mysterious object. For example, it seems not to be known whether or not the strata of $\Sigma_{iness}^{(1)}$ correspond always to "nugatory crossings" (for the definition see e.g. [12]). Moreover, I do not know whether or not the complement of $cl(\Sigma_{iness}^{(1)})$ is still connected in each component of \mathcal{M} .

Conjecture 1 *The two knots shown in Fig.2 are not framed homotopic in \mathcal{M}_{ess} .*

.

But notice that the two knots shown in Fig.3 are framed homotopic in \mathcal{M}_{ess} . The path is shown in the figure too. It uses the fact that the "figure eight" knot can be unknotted both by a positive or a negative crossing change.

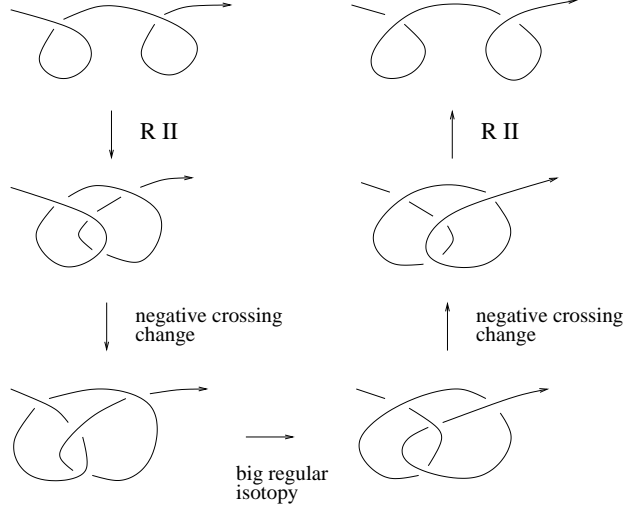


Figure 3: a framed homotopy

So, we do not know much about the components of the essential diagram space besides the fact that each component becomes simply connected (even contractible) if we add again the closure of the non-essential walls $cl(\Sigma_{iness}^{(1)})$.

In this paper we describe a surprising phenomenon: *there are components of the essential diagram space which have non-trivial first homology groups*. This result is far from being obvious and we do not know any other method to prove this. (In finite dimensions this would imply by Alexander duality that $cl(\Sigma_{iness}^{(1)})$ contains a cycle of codimension two in \mathcal{M} and this cycle is not homologically trivial in $cl(\Sigma_{iness}^{(1)})$.)

Our prove uses a 1-cocycle which is constructed in the following way: Let K be a singular knot and let $V_K^s(A, B, C) \in \mathbb{Z}[A, A^{-1}, B, C]$ be the extension of the Jones polynomial for singular links contained in [8] (abusing notation we denote a knot diagram for K by K too). It is defined as follows:

$$V_K^s(A, B, C) = (-A)^{-3w(K)} \langle K \rangle_s \in \mathbb{Z}[A, A^{-1}, B, C]$$

(In [8] we have chosen $C = B^{-1}$ because the polynomial is homogenous in B and C .) The *singular Kauffman bracket* $\langle K \rangle_s$ is defined at crossings as the usual Kauffman bracket (compare [12]) and at double points it is defined as shown in Fig. 4. Here, B and C are new independent variables. Notice that one of the smoothings induces only a piecewise orientation on the link diagram. $w(K)$ is the writhe of the knot diagram K (see e.g. [5]).

Let γ be a generic path in \mathcal{F} which connects a given non singular knot

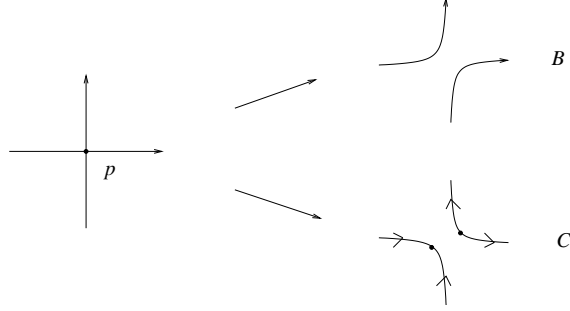


Figure 4: smoothings of a double point

with the unknot. Here, the end points of the paths are allowed to move inside the chambers. A generic homotopy of such a path meets the codimension two part $\Sigma^{(2)}$ of the discriminant in a finite number of points which correspond to knots with exactly two ordinary double points or with exactly one ordinary cusp or with exactly one double point with equal tangent directions (see e.g. [6] and [9]).

We define a *co-orientation* on $\Sigma^{(1)}$ by saying that the positive normal direction corresponds to changing a negative crossing to a positive one. The path γ intersects $\Sigma^{(1)}$ transversally in a finite number of points. Let p be such an intersection point. Abusing notation we denote the corresponding double point by p too and we denote the corresponding singular knot by $K(p)$. Let $ind(p)$ be the intersection index of γ with $\Sigma^{(1)}$ at p .

Definition 2 *Let γ be an oriented path in \mathcal{M}_{ess} which connects a non singular knot K with a diagram of the unknot. The polynomial $Cross(\gamma) \in \mathbb{Z}[A, A^{-1}]$ is defined by the following formula*

$$Cross(\gamma) = \sum_{p \in \gamma \cap \Sigma} ind(p) < K(p) >_s (A, 1, -1).$$

It follows immediately from the definitions that the intersection index of $\gamma \cap \Sigma$ is an invariant of paths γ in \mathcal{M}_{ess} up to homotopy in \mathcal{M}_{ess} , if the end points of the paths are fixed up to regular isotopy. Consequently, the writhe of the unknot at the end of the path γ is completely determined by the writhe $w(K)$ of the knot K and by the intersection index $ind(\gamma) = \gamma \cap \Sigma$. The Whitney index is invariant under crossing changes. (As well known, non singular knots are regularly isotopic if and only if they are isotopic and they share the same writhe and the same Whitney index, compare e.g. [7].) In particular, the intersection index of each loop with $\Sigma^{(1)}$ is zero.

Theorem 1 *Let γ be an oriented loop in \mathcal{M}_{ess} . Then the value of $Cross(\gamma)$ depends only on the homology class of γ . The induced cohomology class $[Cross] \in H^1(\mathcal{M}_{ess}; \mathbb{Z}[A, A^{-1}])$ is non trivial.*

We describe now the first loop γ in \mathcal{M}_{ess} for which $Cross(\gamma)$ is non-trivial (we could not find a simpler example). Let us consider the knots 4_1 and 6_3 (from the Knot Atlas) with their standard diagrams. They are both amphicheiral. Each of them can be unknotted by either a positive or a negative crossing change. Let K be the connected sum $4_1 \# 6_3$. Let h be the homotopy which unknots K by a positive crossing change (i.e. $ind = 1$) of 4_1 and by a negative crossing change (i.e. $ind = -1$) of 6_3 . Let h' be the homotopy which unknots K by a negative crossing change of 4_1 and by a positive crossing change of 6_3 .

A calculation by hand gives the following result:

$$Cross(h \cup (-h')) = (A^{-1} + A + A^{-3} + A^3 + A^{-5} + A^5)(\langle 6_3 \rangle - \langle 4_1 \rangle).$$

Here $\langle . \rangle$ denotes the usual Kauffman bracket. Notice that the diagrams of 4_1 and 6_3 have vanishing writhe. Hence, the Kauffman bracket coincides with the Jones polynomial (see e.g. [12], [11]). We could change K by Reidemeister moves of type I and consider the induced loop $h \cup (-h')$. The value of $Cross(\gamma)$ would change just by a standard factor. This shows that these components of \mathcal{M}_{ess} are non-simply connected too.

The framed long knot K (i.e. the diagram K or like-wise the smooth submanifold $f(\mathbb{R})$) in our example has unknotting number two. The value of $Cross(\gamma)$ looks like the product of some standard polynomial (which depends only on the unknotting number of K) with a linear combination of Jones polynomials of those knots of unknotting number one which are contained in the loop. It seems to be very difficult to prove this in general. But it leads us to the following conjecture.

Conjecture 2 *Let K be a framed long knot of unknotting number one. Let h be a homotopy which unknots K and such that $ind(h) = 1$ (respectively $ind(h) = -1$). Then $Cross(h)$ is invariant under regular isotopy of K .*

Notice that apriori the intersection of the closure of the connected component of K in \mathcal{M}_{ess} with the closure of the component of the unknot could have an infinite number of connected components. Each of these components corresponds to an unknotting of K and they could be all different!

We have verified this conjecture by hand for all diagrams with no more than 6 crossings (which is not a big deal).

Example 1 $Cross(3_1, w = 3, n = 1, ind = -1) = A^{-6} + A^{-4} + A^4$

$$Cross(3_1!, w = 3, n = 1, ind = 1) = -A^{14} - A^{22} - A^{24}$$

$$Cross(4_1, w = 0, n = 0, ind = 1) = A^{-7} + A^{-5} + A + A^3 - A^7$$

$$Cross(4_1, w = 0, n = 0, ind = -1) = A^{-7} - A^{-3} - A^{-1} - A^5 - A^7.$$

Here w is the writhe and n is the Whitney index of the diagram which we unknot. 3_1 is the right trefoil and $3_1!$ is its mirror image, the left trefoil.

Our theorem shows that the situation is more complicated already for framed knots of unknotting number two. The corresponding conjecture would be that $Cross(h)$ is now well defined up to combinations of Jones polynomials of framed knots with unknotting number one.

Remark 1 *There is an extension of the HOMFLY-PT polynomial for singular links by Kauffman and Vogel [14]. A natural idea would be to replace in the construction of $Cross(\gamma)$ the singular Kauffman bracket $\langle K \rangle_s$ by their singular HOMFLY-PT polynomial. Surprisingly, this fails. We can not replace $\langle K \rangle_s$ by our singular Alexander polynomial Δ^s neither (see [8] and the next section).*

Remark 2 *Crossing an inessential wall does not change the knot type but it changes the framing. If we would replace framed isotopy by isotopy then our construction would just lead to a version of the Jones polynomial (as it happened in the first version of the present paper).*

2 Proofs

Let γ be a generic path in the essential diagram space \mathcal{M}_{ess} (i.e. a framed homotopy).

First we observe that $Cross(\gamma)$ is invariant under all homotopies of γ which do not change the intersection with the strata of the discriminant Σ . This comes from the fact that $\langle K(p) \rangle_s (A, 1, -1)$ is an isotopy invariant of framed singular knots (i.e. only Reidemeister moves of type II and III and the additional moves for singular links SII and $SIII$ are allowed, compare [13] and also [8]).

Next we need some simple facts from singularity theory which can be proven with the same methods as e.g. in the Appendix of [9] or [6]. Only the following *accidents* can occur in a generic homotopy of a path γ in \mathcal{M}_{ess} and they can occur only a finite number of times.

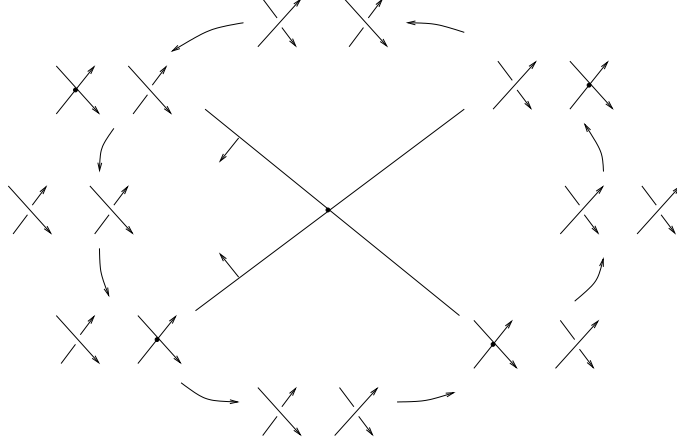


Figure 5: meridian for a pair of double points

I. γ becomes tangential (in an ordinary tangent point) to a stratum of $\Sigma^{(1)}$.

II. γ passes transversally to a stratum of codimension 2, which consists of the transverse intersection of two strata of codimension 1. We denote these strata by $\Sigma_{++}^{(2)}$.

III. γ passes transversally to a stratum of codimension 2, which consists of a (single) double point where the two branches are tangential. We denote these strata by $\Sigma_{tang}^{(2)}$.

IV. γ passes transversally through an ordinary triple point.

All other paths γ in the homotopy are just generic paths in \mathcal{M}_{ess} .

The important point is that the following accident can occur in \mathcal{F} but not in \mathcal{M}_{ess} :

V. γ passes transversally to a stratum of codimension 2, which consists of an ordinary cusp. We denote these strata by $\Sigma_{<}^{(2)}$.

The strata $\Sigma_{<}^{(2)}$ are just (a generic part of) boundaries of inessential walls in \mathcal{F} , but all our paths are in \mathcal{M}_{ess} .

It follows immediately from the definitions that all our polynomials are invariant under accidents of type I.

In order to prove Theorem 1 we have to show that $Cross(m) = 0$ for the meridional loops m of $\Sigma_{++}^{(2)}$ and of $\Sigma_{tang}^{(2)}$.

We show the diagrams for the meridional loop of $\Sigma_{++}^{(2)}$ in Fig. 5. After smoothing the crossings and the double points we are left with exactly the

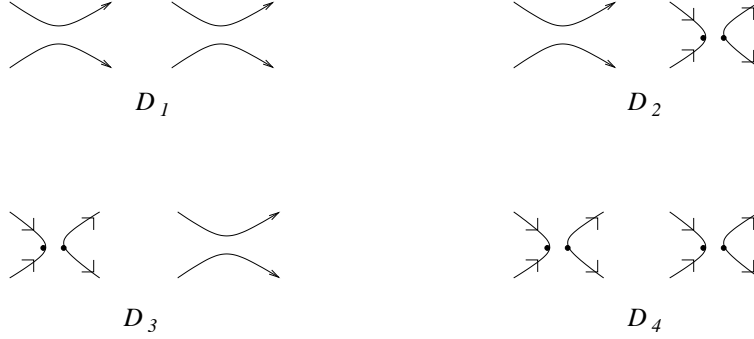


Figure 6: smoothed diagrams near the double points

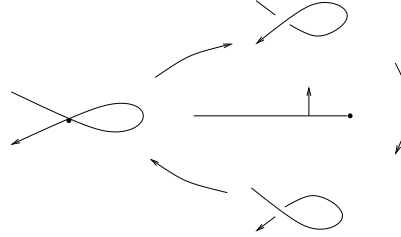


Figure 7: meridian for a cusp

four diagrams D_i shown in Fig. 6. They are in general independent. The meridional loop m of $\Sigma_{++}^{(2)}$ leads to the following equations:

$$\begin{aligned} (-AB - A^{-1}B + AB + A^{-1}B) < D_1 > &= 0 \\ (-A^{-1}B - A^{-1}C + AC + AB) < D_2 > &= 0 \\ (-AC - AB + A^{-1}B + A^{-1}C) < D_3 > &= 0 \\ (-A^{-1}C - AC + A^{-1}C + AC) < D_4 > &= 0. \end{aligned}$$

Here, $< D_i >$ is the usual Kauffman bracket. $Cross(m) = 0$ in general if and only if each coefficient of $< D_i >$ is zero. Therefore we obtain the unique (non trivial) solution

$$C = -B.$$

Each singular link in the framed homotopy has exactly one double point. Therefore we do not loose information by setting $B = 1$, and we obtain exactly the definition of $Cross(\gamma)$.

Notice that it could happen that exactly one of the four strata of $\Sigma^{(1)}$ in Fig. 5 is non essential. In this case we could not push our path γ in \mathcal{M}_{ess} through the corresponding stratum of $\Sigma_{++}^{(2)}$ by a small homotopy. Our theorem implies that sometimes we can not even do it by a big homotopy.

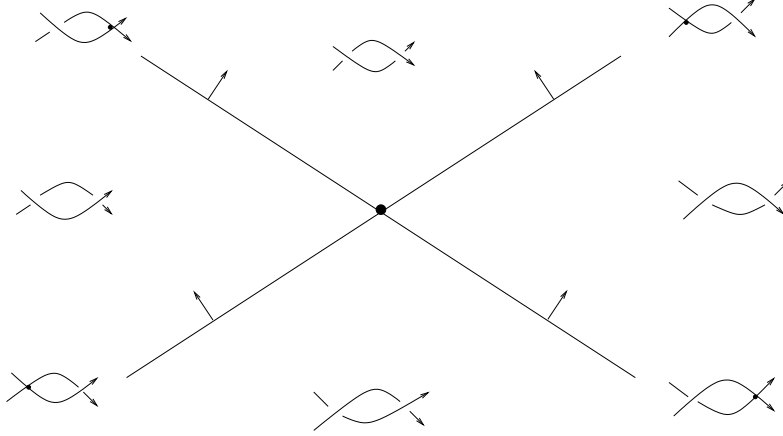


Figure 8: meridian for a double point with equal tangencies

We show the diagrams for the meridional loop of $\Sigma_{<}^{(2)} \subset \mathcal{F}$ in Fig. 7. The value of $Cross(m)$ is a non trivial multiple of a Kauffman bracket. Consequently, $Cross(m)$ is in general non zero.

We show the diagrams for the meridional loop of $\Sigma_{tang}^{(2)}$ in Fig. 8.

$Cross(m) = 0$ follows from the invariance of V^s under the move SII for singular links (compare [8]). (We had to choose in Fig. 8 a type of a Reidemeister II move and orientations on the branches. Taking mirror images or changing orientations of branches leads to the same result.)

The stratum of an ordinary triple point is not smooth. But one easily sees that the meridians form theta-graphs. Using the invariance of $Cross$ under passing $\Sigma_{tang}^{(2)}$ it suffices to prove the invariance for the loops in just one of the theta graphs (compare [7]). The three arcs m_1, m_2, m_3 in the theta-graph are shown in Fig. 10,11 and 12. It suffices hence to prove that $Cross(m_1 \cup (-m_2)) = Cross(m_1 \cup (-m_3)) = 0$.

We show the calculation of $Cross(m_1 \cup (-m_2))$ in Fig. 12. The first equality in Fig. 12 uses twice the fact that $Cross$ is invariant under passing $\Sigma_{++}^{(2)}$. The calculation of $Cross(m_1 \cup (-m_3))$ is completely analogous and is left to the reader.

It is clear that we can replace homotopy by homology in all our constructions because the values are in a commutative ring. We have proven that $Cross(\gamma)$ depends only on the homology class of γ in \mathcal{M}_{ess} . The example in the introduction shows that $Cross$ induces a non trivial cohomology class. The theorem is proven.

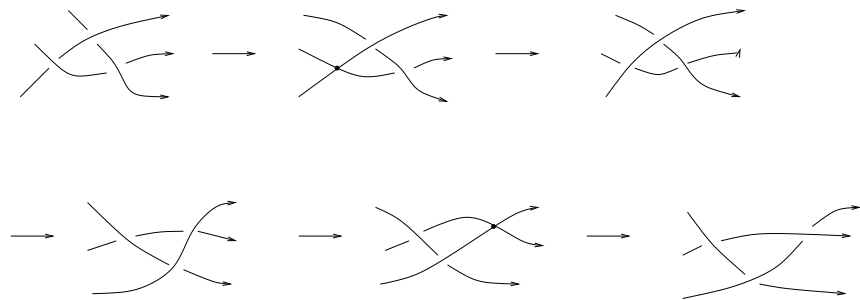


Figure 9: the path m_1

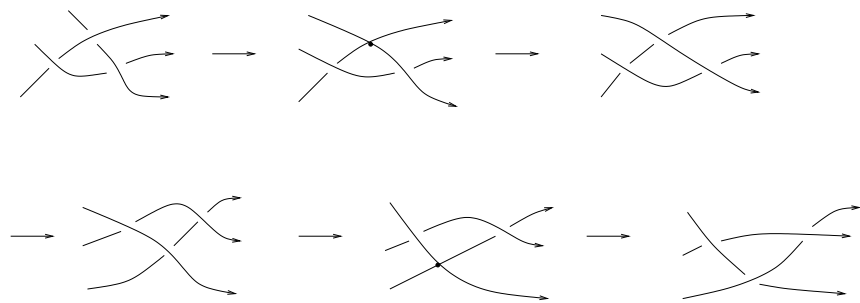


Figure 10: the path m_2

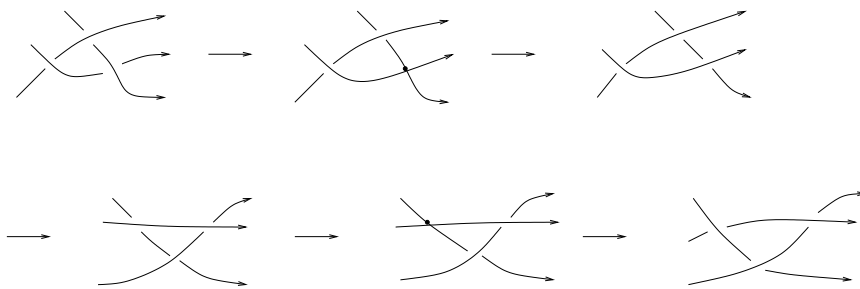


Figure 11: the path m_3

$$\begin{array}{rcccl}
- & \text{[Diagram 1]} & + & \text{[Diagram 2]} & \\
- & \text{[Diagram 3]} & + & \text{[Diagram 4]} & \\
= & \text{[Diagram 5]} & + & \text{[Diagram 6]} & \\
- & \text{[Diagram 7]} & + & \text{[Diagram 8]} & = 0
\end{array}$$

The figure shows a sequence of eight diagrams, each representing a crossing of two strands in a braid-like structure. The strands are labeled with arrows indicating direction. The diagrams are arranged in a grid, with the first three rows showing a sequence of crossings and the fourth row showing the final result as zero. The crossings are labeled with minus signs, plus signs, and an equals sign, indicating a relationship between the diagrams.

Figure 12: $Cross(m_1 \cup (-m_2))$

$$\begin{aligned}
\left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] &= x \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] \\
\left[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right] &= y \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] + \left[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right] \\
\left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] &= a \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right]
\end{aligned}$$

Figure 13: Kauffman-Vogel's skein relation

$$(x - y) \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] = (x - y) \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right]$$

Figure 14: equation from the meridian for a pair of double points

It remains to prove our assertion from Remark 1. We recall the definition of the singular HOMFLY-PT polynomial from [14] in Fig. 13. Here, x, y, a are new independent variables. We want to replace V^s by this polynomial. A calculation for the meridional loop of $\Sigma_{++}^{(2)}$ leads to the equation shown in Fig. 14, which has the unique solution $x = y$. But then the singular HOMFLY-PT polynomial is no longer sensitive for crossing changes of links.

The failure of the singular Alexander polynomial from [8] is also caused by the stratum $\Sigma_{++}^{(2)}$. It leads to the equation $A^4 = 1$, which makes the invariant uninteresting. We leave the verification to the reader.

Acknowledgements— I am grateful to the referee for his useful comments.

References

- [1] Bar-Natan D. : On the Vassiliev knot invariants, Topology 34 (1995) 423-472
- [2] Budney R.,Cohen F. : On the homology of the space of knots, math. GT/0504206

- [3] Budney R., Conant J., Scannell K., Sinha D. : New perspectives on self-linking, *Adv. Math.* 191 (2005) 78-113
- [4] Budney R. : Topology of spaces of knots in dimension 3, *math. GT/0506524*
- [5] Burde G., Zieschang H. : *Knots*, de Gruyter Studies in Mathematics 5, Berlin, (1985)
- [6] Cerf J. : Topologie de certains espaces de plongements, *Bull. S.M.F.* 89 (1961) 227-380
- [7] Fiedler T.: Knot polynomials via one parameter knot theory, *math.GT/0612115*
- [8] Fiedler T.: The Jones and Alexander polynomials for singular links, *arXiv:0706.0084 math.GT*
- [9] Fiedler T., Kurlin V.: A one-parameter approach to knot theory, *math. GT/0606381*
- [10] Hatcher A. : Topological moduli spaces of knots, *math. GT/9909095*
- [11] Jones V.: Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* 126 (1987) 335-388
- [12] Kauffman L. : State models and the Jones polynomial, *Topology* 26 (1987) 395-407
- [13] Kauffman L. : Invariants of graphs in 3-space, *Trans. Amer. Math. Soc.* 311 (1989) 417-465
- [14] Kauffman L., Vogel P. : Link polynomials and a graphical calculus, *J. Knot Theory Ramif.* 1 (1992) 59-104
- [15] Vassiliev V. : Cohomology of knot spaces , *Adv. in Sov. Math. , Theory of Singularities and its Appl. , A.M.S. Providence, R.I.* (1990) , 23-69.
- [16] Vassiliev V. : Combinatorial formulas of cohomology of knot spaces , *Moscow Math. Journal* 1 (2001) 91-123.

Institut de Mathématiques de Toulouse
Université Paul Sabatier et CNRS (UMR 5219)
118, route de Narbonne
31062 Toulouse Cedex 09, France
fiedler@picard.ups-tlse.fr